Foundational Libraries in Naproche

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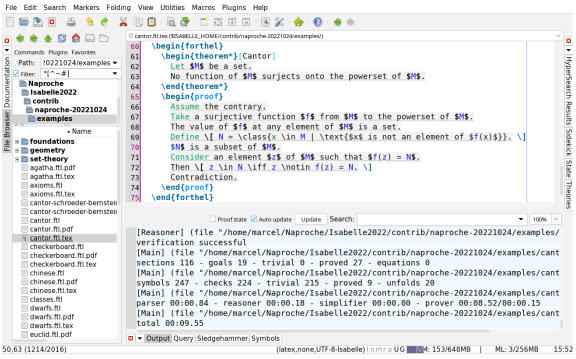
The Naproche System

Naproche = **Na**tural **pro**of **che**cking

- Proof assistant
- Component of Isabelle

ForTheL = **For**mula **The**ory **L**anguage

- Naproche's input language
- Controlled natural language
- LaTeX-compatible



Cantor's Theorem in Isabelle/jEdit

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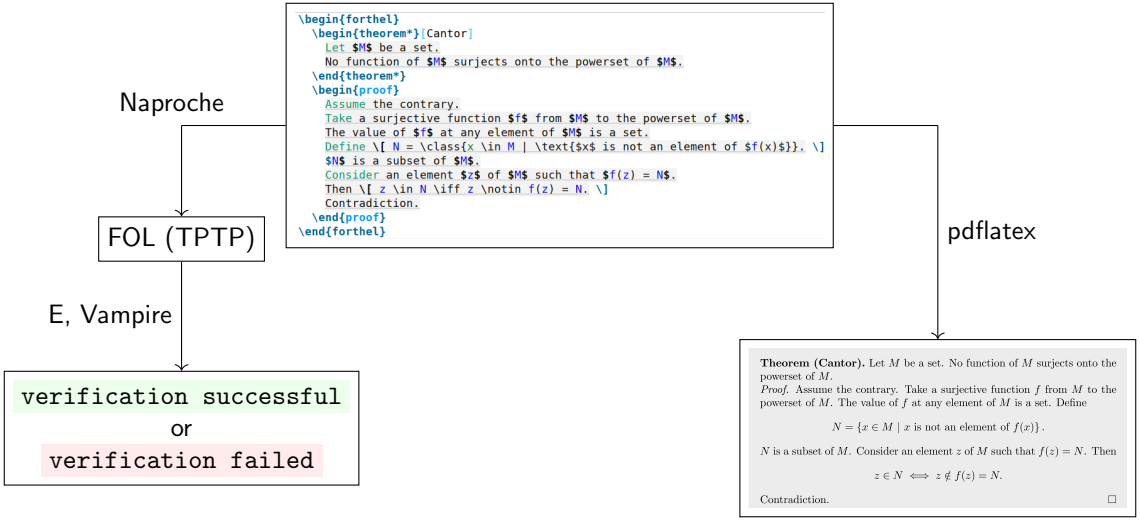
The Naproche System



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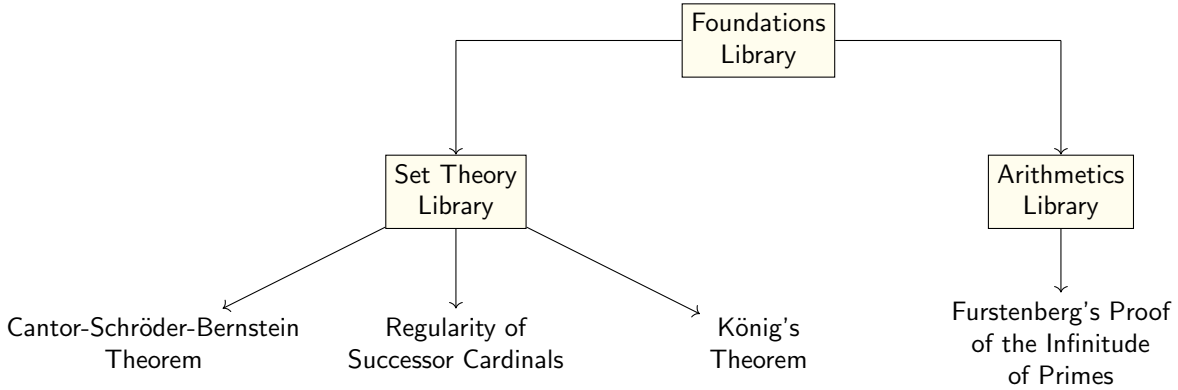
Libraries in Naproche



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Libraries in Naproche

ARITHMETIC_03_3235893452210176

Proposition 3.13. Let n, m, k be natural numbers. Then

$$n + (m + k) = (n + m) + k.$$

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid n + (m + k') = (n + m) + k'\}$.

(1) 0 is contained in Φ. Indeed $n + (m + 0) = n + m = (n + m) + 0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Then $n + (m + k') = (n + m) + k'$. Hence

$$\begin{aligned} & n + (m + (k' + 1)) \\ &= n + ((m + k') + 1) \\ &= (n + (m + k')) + 1 \\ &= ((n + m) + k') + 1 \\ &= (n + m) + (k' + 1). \end{aligned}$$

Thus $k' + 1 \in \Phi$. Qed.

Thus every natural number is an element of Φ. Therefore $n + (m + k) = (n + m) + k$. □

Arithmetics: Proof by induction

FOUNDATIONS_10_1897613305577472

Axiom 10.29 (Choice). Let X be a system of nonempty sets. Then there exists a map f such that $\mathrm{dom}(f) = X$ and $f(x) \in x$ for any $x \in X$.

Foundations: Axiom of choice

SET_THEORY_02_229593678086144

Definition 2.1. An ordinal number is a transitive set α such that every element of α is a transitive set.

Let an ordinal stand for an ordinal number.

SET_THEORY_02_5852994258075648

Definition 2.2. Ord is the class of all ordinals.

Set theory: Definition of ordinal numbers

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The LaTeX Workflow

Libraries are structured as books with a chapter-structure

→ Chapters can be referenced by their file names:

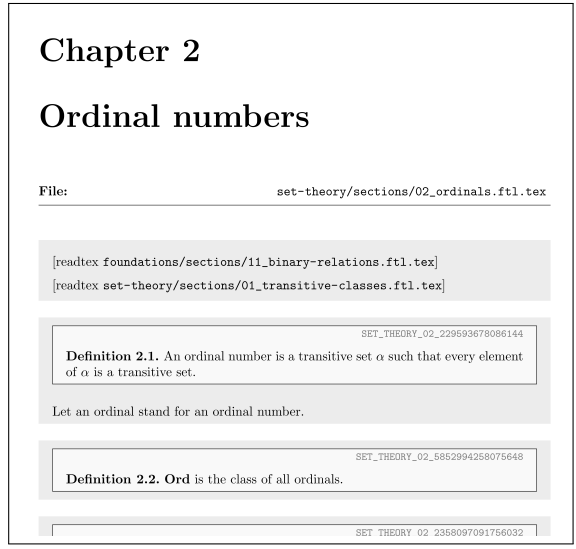
```
set-theory/sections/02_ordinals.ftl.tex
```

→ Chapters depend on each other:

```
[readtex foundations/sections/11_binary-relations.ftl.tex]
```

→ Definitions, theorems etc. can be referenced by unique IDs:

```
SET_THEORY_02_229593678086144
```



Chapter 2

Ordinal numbers

File: set-theory/sections/02_ordinals.ftl.tex

[readtex foundations/sections/11_binary-relations.ftl.tex]
[readtex set-theory/sections/01_transitive-classes.ftl.tex]

SET_THEORY_02_229593678086144

Definition 2.1. An ordinal number is a transitive set α such that every element of α is a transitive set.

Let an ordinal stand for an ordinal number.

SET_THEORY_02_5852994258075648

Definition 2.2. Ord is the class of all ordinals.

SET_THEORY_02_2358097091756032

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The LaTeX Workflow

Referencing statements from libraries:

→ Using the LaTeX package `xr`:

```
\usepackage{xr}
```

→ Specifying a library to reference to:

```
\externaldocument{set-theory/set-theory}
```

→ Using the referencing command `\cref{...}`:

```
\cref{SET_THEORY_06_8113916590686208}
```

We have $|F[\kappa_i]| \leq |\kappa_i|$ (by proposition 6.10).

```
\usepackage{xr}
\externaldocument{set-theory/set-theory}

...

We have $|F[\kappa_i]| \leq |\kappa_i|$
(by \cref{SET_THEORY_06_8113916590686208}).

...
```

Referencing a proposition

SET_THEORY_06_8113916590686208

Proposition 6.10. Let x, y be sets and $f : x \to y$ and $a \subseteq x$. Then $|f[a]| \leq |a|$.

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The Verifying Workflow

Library	Checking time	Definitions/theorems/axioms
Foundations	$\sim$ 10 min.	235
Set Theory	$\sim$ 30 min.	100
Arithmetics	$\sim$ 30 min.	176

$\rightarrow$ **Checking time** does not scale well with the size of a formalization in Naproche

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The Verifying Workflow

Naproche rechecks each library whenever it is imported to another formalization.

→ Annoying **time redundancies**

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The Verifying Workflow

We cannot use different theories in one document.

We cannot use theory morphisms.

We cannot work with instances of theories (i.e. mathematical structures).

→ ForTheL lacks a proper **module system**

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Conclusion & Ideas for Future Work

We have: Both formal *and* human-readable libraries that integrate well in the LaTeX workflow

Current Issues:

Scalability

→ Extending the scope of provers that ForTheL texts can be checked with (e.g. Isabelle, Lean, . . .)

Time redundancy

→ Persistently storing caching results or proof objects

Modularity

→ Adding new language features to ForTheL

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